

A NOTE ON ATIYAH'S Γ -INDEX THEOREM IN HEISENBERG CALCULUS

TATSUKI SETO

ABSTRACT. In this note, we prove an index theorem on Galois coverings for Heisenberg elliptic differential operators, but not elliptic, which is analogous to Atiyah's Γ -index theorem. This note also contains an example of Heisenberg differential operators with a non-trivial Γ -index.

INTRODUCTION

M. F. Atiyah [1] introduced the notion of the Γ -index $\text{index}_\Gamma(\tilde{D})$ for a lifted elliptic differential operator \tilde{D} on a Galois Γ -covering over a closed manifold and proved that the Γ -index $\text{index}_\Gamma(\tilde{D})$ of a lifted operator \tilde{D} equals the Fredholm index $\text{index}(D)$ of the elliptic differential operator D on the base manifold. On the other hand, Atiyah [1] also investigate properties of a Γ -trace tr_Γ at the same time. The Γ -trace is a trace of the Γ -trace operators, so it induces a homomorphism $(\text{tr}_\Gamma)_*$ from K_0 -group of the Γ -compact operators to the real numbers. Out of a lifted elliptic differential operator \tilde{D} , we can define the Γ -index class $\text{Ind}_\Gamma(\tilde{D})$ by using the Connes-Skandalis idempotent [3, II.9.α (p.131)] and send it by the induced homomorphism $(\text{tr}_\Gamma)_*$, then the image $(\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{D}))$ equals the Γ -index $\text{index}_\Gamma(\tilde{D})$ and thus the Fredholm index $\text{index}(D)$ of the elliptic differential operator D on the base manifold.

On the other hand, there is another pseudo-differential calculus on Heisenberg manifolds which is called the Heisenberg calculus; see, for instance [5]. Roughly speaking, Heisenberg calculus is “weighted” calculus and the product of the “Heisenberg principal symbols” are defined by convolution product. When the Heisenberg principal symbol of P is invertible, we call P a Heisenberg elliptic operator. Note that any Heisenberg elliptic operator is not elliptic. For a Heisenberg elliptic operator P , we can construct a parametrix by using its inverse, so P is a Fredholm operator if the base manifold is closed. Thus the Fredholm index of P on a closed manifold is well defined, but a solution of an index problem of P does not obtained in general. However, index problems for Heisenberg elliptic operators on contact manifolds or foliated manifolds are solved by E. van Erp and P. F. Baum; see [2], [6], [7], [9].

In this note, we study that we can define the Γ -index $\text{index}_\Gamma(\tilde{P})$ and the Γ -index class $\text{Ind}_\Gamma(\tilde{P})$ for a lifted Heisenberg elliptic differential operator \tilde{P} by using a parametrix. Once these ingredients are defined, the proof of the matching of three ingredients $\text{index}_\Gamma(\tilde{P})$, $(\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{P}))$ and $\text{index}(P)$, is straightforward; see

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subsection 2.1. We also investigate an example of Heisenberg differential operators on a contact manifold with non-trivial Γ -index by using the index formula in [2]; see subsection 2.2.

1. SHORT REVIEW OF ATIYAH'S Γ -INDEX THEOREM

In this section, we recall Atiyah's Γ -index theorem in ordinary pseudo-differential calculus. The main reference of this section is Atiyah's paper [1]. Let $\widetilde{M} \rightarrow M$ be a Galois covering with a deck transformation group Γ over a closed manifold M with a smooth measure μ and $D : C^\infty(E) \rightarrow C^\infty(F)$ an elliptic differential operator on Hermitian vector bundles $E, F \rightarrow M$. We lift these ingredients on \widetilde{M} and denote by $\widetilde{D} : C^\infty(\widetilde{E}) \rightarrow C^\infty(\widetilde{F})$ and $\widetilde{\mu}$. Let $\text{Ker}_{L^2}(\widetilde{D})$ (resp. $\text{Ker}_{L^2}(\widetilde{D}^*)$) be the L^2 -solutions of $\widetilde{D}u = 0$ (resp. $\widetilde{D}^*u = 0$) and denote by Π_0 (resp. Π_1) the orthogonal projection on a closed subspace $\text{Ker}_{L^2}(\widetilde{D})$ (resp. $\text{Ker}_{L^2}(\widetilde{D}^*)$) of the L^2 -sections.

A Γ -invariant bounded operator T on the L^2 -sections $L^2(\widetilde{E})$ of \widetilde{E} is of Γ -trace class if $\phi T \psi \in L^2(\widetilde{E})$ is of trace class for any compactly supported smooth functions ϕ, ψ on \widetilde{M} . Denote by \mathcal{L}_Γ^1 the set of Γ -trace class operators and $\text{tr}_\Gamma(T)$ the Γ -trace of a Γ -trace class operator T defined by

$$\text{tr}_\Gamma(T) = \text{Tr}(\phi T \psi) \in \mathbb{C}.$$

Here, the right hand side is the trace of a trace class operator $\phi T \psi$ and this quantity does not depend on the choice of functions ϕ, ψ . By using ellipticity of \widetilde{D} , operators $\phi \Pi_0 \psi$ and $\phi \Pi_1 \psi$ are smoothing operators on compact sets. Thus Π_0 and Π_1 are of Γ -trace class and thus one obtains the Γ -index of \widetilde{D} :

$$\text{index}_\Gamma(\widetilde{D}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) \in \mathbb{R}.$$

In the context of the Γ -index theorem, the most important class of Γ -trace class operators is the lifts of almost local smoothing operators on M . Let S be an almost local smoothing operator with a smooth kernel k_S and \widetilde{S} a lift of S . Then \widetilde{S} is of Γ -trace class and its Γ -trace is calculated by the following:

$$(*) \quad \text{tr}_\Gamma(\widetilde{S}) = \int_M \text{tr}(k_S(x, x)) d\mu = \text{Tr}(S).$$

Denote by \mathcal{K}_Γ the C^* -closure of \mathcal{L}_Γ^1 and $K_0(\mathcal{K}_\Gamma)$ the analytic K_0 -group. Then tr_Γ induces a homomorphism of abelian groups by substitution:

$$(\text{tr}_\Gamma)_* : K_0(\mathcal{K}_\Gamma) \rightarrow \mathbb{R}.$$

On the other hand, since D is an elliptic differential operator, there exist an almost local parametrix Q and almost local smoothing operators S_0, S_1 such that one has $QD = 1 - S_0$ and $DQ = 1 - S_1$. Denote by $\widetilde{Q}, \widetilde{S}_0$ and \widetilde{S}_1 lifts of these operators and then one has same relations $\widetilde{Q}\widetilde{D} = 1 - \widetilde{S}_0$ and $\widetilde{D}\widetilde{Q} = 1 - \widetilde{S}_1$. Set

$$e_{\widetilde{D}} = \begin{bmatrix} \widetilde{S}_0^2 & \widetilde{S}_0(1 + \widetilde{S}_0)\widetilde{Q} \\ \widetilde{S}_1\widetilde{D} & 1 - \widetilde{S}_1^2 \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By $\widetilde{Q}\widetilde{S}_1 = \widetilde{S}_0\widetilde{Q}$ and $\widetilde{S}_1\widetilde{D} = \widetilde{D}\widetilde{S}_0$, one has $e_{\widetilde{D}}^2 = e_{\widetilde{D}}$, that is, $e_{\widetilde{D}}$ is an idempotent. Note that this idempotent $e_{\widetilde{D}}$ is called the Connes-Skandalis idempotent; see, for

instance [3, II.9.α (p.131)]. Moreover, a difference $e_{\tilde{D}} - e_1$ is of Γ -trace class. Hence we can define a Γ -index class

$$\text{Ind}_\Gamma(\tilde{D}) = [e_{\tilde{D}}] - [e_1] \in K_0(\mathcal{K}_\Gamma).$$

By the definition of a map $(\text{tr}_\Gamma)_*$ and Atiyah's paper, one has the following:

Theorem 1.1 (Atiyah's Γ -index theorem [1, Theorem 3.8]). *In the above settings, we have the following equality:*

$$\text{index}_\Gamma(\tilde{D}) = (\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{D})) = \text{index}(D) \in \mathbb{Z}.$$

As described in subsection 2.1, Atiyah's proof of matching of these ingredients in the above equality does not essentially use ellipticity. Note that ellipticity of D and \tilde{D} is only used in the definition of these ingredients.

2. ATIYAH'S Γ -INDEX THEOREM IN HEISENBERG CALCULUS

Let (M, H) be a closed Heisenberg manifold, that is, M is a closed manifold and $H \subset TM$ is a hyperplane bundle. Let $P : C^\infty(E) \rightarrow C^\infty(F)$ be a Heisenberg elliptic differential operator on Hermitian vector bundles $E, F \rightarrow (M, H)$, that is, the Heisenberg principal symbol $\sigma_H(P)$ of P is an invertible element. In this section, we prove the Γ -index theorem for P , which is analogous to Atiyah's Γ -index theorem. Note that P is not an elliptic operator in the sense of ordinary pseudo-differential calculus.

2.1. Statement and proof. By [5, Proposition 3.3.1], there exist parametrix Q and smoothing operators S_0, S_1 such that one has $QP = 1 - S_0$ and $PQ = 1 - S_1$. Thus P is a Fredholm operator and one has the Fredholm index $\text{index}(P) \in \mathbb{Z}$ of P by compactness of M . Moreover, since a integral kernel of Q is smooth off the diagonal, we can choose Q as an almost local operator and then S_0 and S_1 are also almost local operators.

Let $\tilde{M} \rightarrow M$ be a Galois covering with a deck transformation group Γ over a closed manifold M with a smooth measure μ . We lift all structures on M to \tilde{M} . Then (\tilde{M}, \tilde{H}) is a Heisenberg manifold, $\tilde{P} : C^\infty(\tilde{E}) \rightarrow C^\infty(\tilde{F})$ is a Heisenberg elliptic differential operator and one has $\tilde{Q}\tilde{P} = 1 - \tilde{S}_0$ and $\tilde{P}\tilde{Q} = 1 - \tilde{S}_1$.

Since P is a differential operator (in particular, P is local), there exists a constant $C = C(\tilde{P}, \phi) > 0$ such that we have an inequality

$$\|\tilde{P}(\phi f)\|_{L^2} \leq C(\|\chi \tilde{P}f\|_{L^2} + \|\chi f\|_{L^2})$$

for any $f \in C^\infty(\tilde{E})$; see [5, Proposition 3.3.2]. Here, $\phi, \chi \in C_c^\infty(\tilde{M})$ are compactly supported smooth functions and one assumes $\chi = 1$ on the support of ϕ . Thus by using Atiyah's technique of the proof of [1, Proposition 3.1], we have the following:

Lemma 2.1. *The minimal domain of \tilde{P} equals the maximal domain of \tilde{P} .*

By Lemma 2.1, \tilde{P} has the unique closed extension denoted by the same letter \tilde{P} and thus the closure of the formal adjoint of \tilde{P} (the formal adjoint is also Heisenberg elliptic) equals the Hilbert space adjoint \tilde{P}^* .

On the other hand, any L^2 -solutions of $\tilde{P}u = 0$ and $\tilde{P}^*u = 0$ are smooth by existence of a parametrix. Thus the orthogonal projection Π_0 (resp. Π_1) onto a closed subspace $\text{Ker}_{L^2}(\tilde{P})$ (resp. $\text{Ker}_{L^2}(\tilde{P}^*)$) of the L^2 -sections is of Γ -trace class

since operators $\phi\Pi_0\psi$ and $\phi\Pi_1\psi$ are smoothing operators on compact sets. Thus one obtains the well-defined Γ -index of \tilde{P} .

Definition 2.2. *The Γ -index of \tilde{P} is defined to be*

$$\text{index}_\Gamma(\tilde{P}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) \in \mathbb{R}.$$

By using operators $\tilde{P}, \tilde{Q}, \tilde{S}_0$ and \tilde{S}_1 , we define

$$e_{\tilde{P}} = \begin{bmatrix} \tilde{S}_0^2 & \tilde{S}_0(1 + \tilde{S}_0)\tilde{Q} \\ \tilde{S}_1\tilde{P} & 1 - \tilde{S}_1^2 \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since a difference $e_{\tilde{P}} - e_1$ is of Γ -trace class, one can define a Γ -index class of \tilde{P} .

Definition 2.3. *We define Γ -index class $\text{Ind}_\Gamma(\tilde{P})$ of \tilde{P} by*

$$\text{Ind}_\Gamma(\tilde{P}) = [e_{\tilde{P}}] - [e_1] \in K_0(\mathcal{K}_\Gamma).$$

By using a Γ -trace, we have the Γ -index theorem in Heisenberg calculus.

Theorem 2.4. *Let P be a Heisenberg elliptic differential operator on a closed Heisenberg manifold (M, H) and \tilde{P} its lift as previously. Then one has*

$$\text{index}_\Gamma(\tilde{P}) = (\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{P})) = \text{index}(P) \in \mathbb{Z}.$$

Proof. First, note that equalities

$$\begin{aligned} 1 - S_0^2 &= 1 - (1 - QP)^2 = (2Q - QPQ)P \quad \text{and} \\ 1 - S_1^2 &= 1 - (1 - PQ)^2 = P(2Q - QPQ), \end{aligned}$$

and note that operators $2Q - QPQ$, S_0^2 and S_1^2 are almost local operators. Thus by Atiyah's technique in [1, Section 5], one has

$$\text{index}_\Gamma(\tilde{P}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) = \text{tr}_\Gamma(\tilde{S}_0^2) - \text{tr}_\Gamma(\tilde{S}_1^2).$$

Next, the definition of a map $(\text{tr}_\Gamma)_*$, one has

$$(\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{P})) = \text{tr}_\Gamma \begin{bmatrix} \tilde{S}_0^2 & \tilde{S}_0(1 + \tilde{S}_0)\tilde{Q} \\ \tilde{S}_1\tilde{P} & -\tilde{S}_1^2 \end{bmatrix} = \text{tr}_\Gamma(\tilde{S}_0^2) - \text{tr}_\Gamma(\tilde{S}_1^2).$$

Since operators \tilde{S}_0^2 and \tilde{S}_1^2 are lifts of almost local smoothing operators, one has

$$\text{index}(P) = \text{Tr}(S_0^2) - \text{Tr}(S_1^2) = \text{tr}_\Gamma(\tilde{S}_0^2) - \text{tr}_\Gamma(\tilde{S}_1^2)$$

by using $(*)$ in Section 1. This proves the equality in the theorem. \square

Remark 2.5. *As pointed out in [8, Section 4], the results in [5, Section 3] hold verbatim for arbitrary codimension. That is, we do not need to assume that a distribution H is of codimension 1.*

2.2. Example. Index problems for Heisenberg elliptic operators on arbitrary closed Heisenberg manifolds are not solved yet. However, van Erp [6, 7] and Baum and van Erp [2] solved the index problem on contact manifolds, which are good examples of Heisenberg manifolds. In this subsection, we investigate an example of Heisenberg elliptic differential operators with non-trivial Γ -index on a Galois covering over a closed contact manifold by using the index formula for subLaplacians.

Let $T^2 = S^1 \times S^1 = \{(e^{ix}, e^{iy})\}$ be a 2-torus and set

$$e(x, y) = \begin{bmatrix} f(x) & g(x) + h(x)e^{iy} \\ g(x) + h(x)e^{-iy} & 1 - f(x) \end{bmatrix}.$$

Here, let f be a $[0, 1]$ -valued $2\pi\mathbb{Z}$ -periodic function on \mathbb{R} such that $f(0) = 1$ and $f(\pi) = 0$ and set $g(x) = \chi_{[0, \pi]}(x)\sqrt{f(x) - f(x)^2}$ and $h(x) = \chi_{[\pi, 2\pi]}(x)\sqrt{f(x) - f(x)^2}$; see [4, Section I. 2]. Moreover, we assume f , g and h are smooth functions. Then e defines an $M_2(\mathbb{C})$ -valued smooth function on T^2 .

Since e is an idempotent of rank 1, e defines an complex line bundle E on T^2 . As well known, the first Chern class $c_1(E)$ of E is given by a 2-form

$$\frac{-1}{2\pi i} \text{tr}(e(de)^2) = \frac{-1}{\pi} (hh' + 2f'h^2 - 2fhh') dx \wedge dy.$$

Thus, by using an equality $h^2 = f - f^2$ on $[\pi, 2\pi]$, we can calculate the first Chern number of E :

$$\int_{T^2} c_1(E) = - \int_{\pi}^{2\pi} f' dx = -1.$$

Let $T^3 = T^2 \times S^1 = \{(e^{ix}, e^{iy}, e^{iz})\}$ be a 3-torus and $q : T^3 \rightarrow T^2$ the projection onto T^2 of the first component. Set $\theta_k = \cos(kz)dx - \sin(kz)dy$ for a positive integer k , $H_k = \ker(\theta_k)$, $f_l(x, y, z) = e^{ilz} + 1$ for a integer l and $F = q^*E$. Then (T^3, H_k) is a contact manifold and H_k is a flat vector bundle. Denote by T_k the Reeb vector field for θ_k and $\Delta_{H_k}^F = -\nabla_{X_k}^F \nabla_{X_k}^F - \nabla_{Y_k}^F \nabla_{Y_k}^F$ the sum of squares on F , where $\{X_k, Y_k\}$ is a local frame of H_k . Set

$$P_{k,l} = \Delta_{H_k}^F + if_l \nabla_{T_k}^F.$$

Since the values of $f_l - n$ contained in \mathbb{C}^\times for any odd integer n , an operator $P_{k,l} : C^\infty(F) \rightarrow C^\infty(F)$ is a Heisenberg elliptic differential operator of Heisenberg order 2. By the index formula for $P_{k,l}$ in [2, Example 6.5.3], one has

$$\text{index}(P_{k,l}) = \int_{T^3} \frac{-1}{2\pi i} e^{-ilz} de^{ilz} \wedge c_1(F) = \frac{-1}{2\pi i} \int_{S^1} e^{-ilz} de^{ilz} \int_{T^2} c_1(E) = l.$$

Note that a contact structure H_k is a lift of H_1 by a k -fold cover $p_k : T^3 \rightarrow T^3$; $(e^{ix}, e^{iy}, e^{iz}) \mapsto (e^{ix}, e^{iy}, e^{ikz})$. Since the lift $\widetilde{P_{1,l}}$ of a subLaplacian $P_{1,l}$ by p_k equals $P_{k,kl}$, we have the $\Gamma(= \mathbb{Z}/k\mathbb{Z})$ -index of $\widetilde{P_{1,l}}$:

$$\text{index}_\Gamma(\widetilde{P_{1,l}}) = \frac{1}{k} \text{index}(\widetilde{P_{1,l}}) = \frac{1}{k} \text{index}(P_{k,kl}) = l = \text{index}(P_{1,l}).$$

Next, we consider a general Galois covering of T^3 . Let $X \rightarrow T^3$ be a Galois covering with a deck transformation group Γ , which is a quotient of $\pi_1(T^3) = \mathbb{Z}^3$, for example, $X = \mathbb{R}^3$ and $\Gamma = \mathbb{Z}^3$ the universal covering. By Theorem 2.4, we have a non-trivial Γ -index as follows:

$$\text{index}_\Gamma(\widetilde{P_{k,l}}) = \text{index}(P_{k,l}) = l.$$

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GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FUROCHO, CHIKUSAKU, NAGOYA,
JAPAN

E-mail address: m11034y@math.nagoya-u.ac.jp